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ON THE SOLUTION OF PROBLEMS OF ATMOSPHERIC MOTION BY MEANS OF MODEL EXPERIMENTS

CARL GUSTAF ROSSBY, Statens Meteorologisk-Hydrografiska Anstalt, Stockholm, Sweden

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The difficulties which arise when we try to integrate the fundamental equations of hydrodynamics, and which are due to their complicated, nonlinear form, have long since forced branches of this science to abandon the classical theoretical lines of work for new and more fertile practical methods. The impossibility of expressing in a satisfactory mathematical way the laws and the influences of the disturbing phenomena which we generally include under the name of turbulence, has doubtless contributed to this change. In naval architecture and in aerodynamics, experimental methods have been used for many years. I need only to remind the reader of Eiffel's famous experiments for determinating the resistance of air against differently shaped bodies. Through the invention of airships and airplanes these methods have gained an increased importance. The circulation and pressure distribution around wings and propellers are nowadays determined by means of model experiments in wind tunnels; from these observations the corresponding data for full scale airplanes are obtained through an easy computation.

The possibility of this full scale computation depends entirely upon the existence of *similar motions*. Following closely the methods of Bairstow (*1*), I shall here try to develop the fundamental conception of *dynamic similarity*.

The movement of a fluid within a given space (A') can be described by the three hydrodynamical equations (supplemented by the equation of continuity and certain boundary conditions). The first of these equations has the following form in the case of the motion of an incompressible fluid under no external forces referred to an inertial frame:

$$\rho'\frac{du'}{dt'}=-\frac{\delta p'}{\delta x'}+\mu'\left[\frac{\delta^2 u'}{\delta x'^2}+\frac{\delta^2 u'}{\delta y'^2}+\frac{\delta^2 u'}{\delta z'^2}\right] \tag{1}$$

Here ρ' means the density, p' the pressure, μ' the viscosity, u', v', and w' the components of velocity along the axes of x', y', and z'; t' is the time.

For $\frac{du'}{dt'}$ we have the expression

$$\frac{du'}{dt'}=\frac{\delta u'}{\delta t'}+u'\,\frac{\delta u'}{\delta x'}+v'\,\frac{\delta u'}{\delta y'}+w'\,\frac{\delta u'}{\delta z'}$$

Thus $\frac{du'}{dt'}$ means the x'–component of acceleration of an individual fluid element.

We now demand similar movement of another liquid in a similar space (A), the linear dimensions of which are $\frac{1}{L}$ of the dimensions of A'. By similarity we then mean, that corresponding to any of the instantaneous configurations of stream lines and isobaric surfaces in (A') there could be found a similar configuration in (A). The sequence of corresponding states of motion generally runs at different rates in the two spaces; the ratio between corresponding times in (A') and (A) or the time scale may be called T.

The equations of motion for the space A have the form

$$\rho\frac{du}{dt}=-\frac{\delta p}{\delta x}+\mu\left[\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}+\frac{\delta^2 u}{\delta z^2}\right], \tag{2}$$

where notations similar to those in equation (1) are used. With the above given values for length and time scales, we easily obtain the following relations between corresponding quantities in A' and A:

$$\begin{gathered} dx'=Ldx,\ dy'=Ldy,\ dz'=Ldz,\ dt'=Tdt \\ u'=\frac{L}{T}u,\ \frac{du'}{dt'}=\frac{L}{T^2}\frac{du}{dt},\ \frac{\delta^2 u'}{\delta x'^2}=\frac{1}{LT}\frac{\delta^2 u}{\delta x^2} \end{gathered} \tag{3}$$

Introducing the kinematic coefficients of viscosity, $\nu'=\frac{\mu'}{\rho'}$ and $\nu=\frac{\mu}{\rho}$, we see that (1) can be written in the form

$$\frac{L}{T^2}\frac{du}{dt}=-\frac{\rho}{\rho'}\frac{dp'}{dp}\cdot\frac{1}{L}\frac{1}{\rho}\frac{\delta p}{\delta x}+\frac{\nu'}{\nu}\cdot\frac{\nu}{LT}\left[\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}+\frac{\delta^2 u}{\delta z^2}\right]$$

or

$$\frac{du}{dt}=-\frac{\rho T^2}{\rho' L^2}\frac{dp'}{dp}\frac{1}{\rho}\frac{\delta p}{\delta x}+\frac{T\nu'}{L^2\nu}\cdot\nu\left[\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}+\frac{\delta^2 u}{\delta z^2}\right] \tag{1b}$$

If now the two motions are similar, the equations (1b) and (2) must be identical; thus we obtain the following necessary conditions

$$\text{(4a)}\quad \frac{\nu'}{\nu}\frac{T}{L^2}=1;\quad \text{(4b)}\quad \frac{\rho}{\rho'}\frac{T^2}{L^2}\frac{dp'}{dp}=1$$

Let us denote by U' a characteristic velocity and by D' a characteristic length in A'; letting U and D be the corresponding quantities in A. The condition (4a) can then be written

$$\frac{U'D'}{\nu'}=\frac{UD}{\nu} \tag{5}$$

$\frac{U'D'}{\nu'}$ is a nondimensional pure number, which generally is denoted as Reynold's number. A condition necessary for dynamical similarity is therefore, that the two systems should have the same Reynold's number.

If we suppose the two liquids and the length scale L to be given, the condition (4a) will, obviously, fix the time scale.

The ratio between corresponding pressure differences in (A') and (A) can be obtained from (4b), which may be written

$$\frac{dp'}{dp}=\frac{\rho'}{\rho}\frac{L^2}{T^2} \tag{6a}$$

or

$$\frac{dp'}{dp}=\frac{\rho' U'^2}{\rho U^2} \tag{6b}$$

The equation (6b) does not involve a new condition but could easily be derived through a consideration of dimensions.

From the form of Reynold's number a conclusion can be drawn, which later will prove to be of a certain importance. If we magnify the linear dimensions of a system, the Reynold's number will change in the same way as if we had kept the dimensions constant but diminished the coefficient of viscosity. We may therefore conclude that the influence of internal friction more and more decreases with increasing dimensions of the system considered.

Until now we have considered motions which take place solely under the influence of internal forces, viz, pressure gradients and friction. Atmospheric movements are however essentially determined by the action of gravity (g). If this force be introduced we obtain a new relation between L and T. The equation for the motion along the vertical axis takes the form

$$\frac{dw'}{dt'}=-\frac{1}{\rho'}\frac{\delta p'}{\delta z'}-g+\nu'\left[\frac{\delta^2 w'}{\delta x'^2}+\frac{\delta^2 w'}{\delta y'^2}+\frac{\delta^2 w'}{\delta z'^2}\right] \tag{7a}$$

or

$$\frac{Ldw}{T^2dt}=-\frac{\rho}{L\rho'}\frac{dp'}{dp}\cdot\frac{1}{\rho}\frac{\delta p}{\delta z}-g+\frac{\nu'}{\nu}\frac{\nu}{LT}\left[\frac{\delta^2 w}{\delta x^2}+\frac{\delta^2 w}{\delta y^2}+\frac{\delta^2 w}{\delta z^2}\right] \tag{7b}$$

After multiplication with $\frac{T^2}{L}$ and reductions by means of (4a) and (4b) the equation (7b) can be reduced to

$$\frac{dw}{dt}=-\frac{1}{\rho}\frac{\delta p}{\delta z}-\frac{T^2}{L}g+\nu\left[\frac{\delta^2 w}{\delta x^2}+\frac{\delta^2 w}{\delta x^2}+\frac{\delta^2 w}{\delta z^2}\right] \tag{7c}$$

The corresponding equation for the space A can be written

$$\frac{dw}{dt}=-\frac{1}{\rho}\frac{\delta p}{\delta z}-g+\nu\left[\frac{\delta^2 w}{\delta x^2}+\frac{\delta^2 w}{\delta y^2}+\frac{\delta^2 w}{\delta z^2}\right] \tag{8}$$

Comparing (7c) and (8), which must be identical in the case of similarity, we obtain as a necessary condition

$$L=T^2 \tag{9a}$$

or

$$\frac{U'^2}{D}=\frac{U^2}{D} \tag{9b}$$

Combining (9a) with (4a) we see that our liberty in the choice of scales and model fluid has now become very restricted. If the model fluid is given, the conditions (4a) and (9a) will fix the two scales.

It may now be questioned whether the introduction of the deviating force of the earth's rotation will lead to any new necessary conditions. The answer is easily obtained. This force is proportional to the angular velocity (Ω') of the earth at the latitude considered. To obtain similarity we must obviously give the model vessel a certain angular velocity, which may be denoted by Ω. Now, Ω' and Ω have the dimensions of an inverse time; they must therefore fulfill the relation

$$\Omega'=\frac{1}{T}\Omega \tag{10}$$

However, since we are able within a wide range to give the vessel arbitrary angular velocities, this condition will not restrict our choice of scales and model fluid.

The practical value of the conception of dynamic similarity obviously lies in the fact that we in some cases may be able to observe and measure, in a model experiment, quantities which in the original space A' are beyond our reach. Multiplying the observed data by certain powers of $\frac{\rho'}{\rho}$, L, and T, according to the dimensions of the quantities considered, we can then easily obtain the required values for the original space, provided the necessary conditions (4a) and (9a) are fulfilled.

Suppose now, that we have observed a certain synoptic distribution of density and velocity in the atmosphere. As already pointed out, it is generally impossible to compute mathematically the movements which will develop from this original state. However, if we are able to imitate in a model (A) the observed initial conditions, this space will act as a kind of mechanical integrator. If we then study and measure the model movements by means of a motion-picture camera, certain computations will give us the corresponding states of motion in the atmosphere.

In the attempt to apply this method to meteorological problems, several serious difficulties will arise. Most atmospheric movements take place under simultaneous gain and loss of heat, and these thermal processes often have a marked influence upon the state of motion. The above suggested method would therefore require the extension of the conception of similarity also to the thermal phenomena, which would be impossible.

Even if we limit our study to pure adiabatic movements, the compressibility of the air will introduce a new condition necessary for dynamic similarity. However, in the large atmospheric movements, which are principally horizontal, the influence of compressibility can generally be neglected; in which case the new condition may be disregarded.

Proceeding now under this assumption, the air as studied by us is reduced to an incompressible fluid. Provided the two conditions (4a) and (9a) are fulfilled, we then ought to be able to use model experiments in the study of this atmosphere. However, our liberty in the choice of model fluid and scales is so restricted by these two conditions, that the experiments become practically impossible. We will therefore limit our study to problems in which the atmosphere to a first approximation may be regarded as an ideal, nonviscous fluid. It has been pointed out above that the influence of viscosity will increase as the dimensions of the system decrease. Thus, even if this assumption be justified for the atmosphere, it is a question whether it will apply to the model movements; a satisfactory answer can probably be obtained only from the experiments themselves. It will obviously be necessary to use for the experiments liquids which are as fluent as possible. In any case the viscosity will play a much greater rôle in the model (A) than in reality; no numerical conclusions as to the rate of dissipation of kinetic energy could, therefore, be drawn from measurements in (A).

We have now reduced the atmosphere to an ideal, incompressible liquid. Of the conditions for dynamic similarity only one remains, viz:

$$L=T^2$$

Suppose now, that the movements which we intend to study extend over a horizontal area of about 2,000 km. in diameter, i. e., about the distance between Key West and New York. In vertical direction they reach the upper limit of the troposphere, i. e., about 10 km. high. If we construct a model of this space, using a length scale of $L=10^6$, this portion of the atmosphere would correspond to a circular liquid layer of only 1 cm. thickness but with 2 m. diameter. In order to imitate air masses of different temperature we have to use nonmiscible liquids of slightly different densities. Due to the small thickness of the liquid layers considered, their movements will become strongly influenced by surface tensions and irregularities at the bottom of the model vessel. Obviously the small height of the atmosphere compared with its horizontal extension acts as a great obstacle against the use of model experiments in dynamical meteorology.

To avoid this difficulty we will now assume, that the vertical velocities and accelerations may be neglected in the dynamical equations. In most large-scale atmospheric movements this assumption is justified. The equations of motion in (A') will then take the following form:

$$\text{(11)}\qquad \begin{aligned} \frac{\rho' du'}{dt'} &= -\frac{\delta p'}{\delta x'} \\ \rho' g &= -\frac{\delta p'}{\delta z'} \end{aligned}$$

We now seek a corresponding state of motion in a model space (A). This space is obtained by using one length scale, L, for horizontal distances, and another, l, for the vertical, the time scale, as before, being T. Thus we get:

$$\text{(12)}\qquad dx'=Ldx,\ dy'=Ldy,\ dz'=ldz,\ dt'=Tdt$$

We will now show, that under the above assumption (neglecting the vertical velocities and accelerations in the equations of motion) dynamical similarity may be obtained, provided one necessary condition is fulfilled. By means of (12) the equations (11) can be transformed into:

$$\text{(11b)}\qquad \begin{aligned} \frac{L}{T^2}\frac{\rho'}{\rho}\cdot\rho\cdot\frac{du}{dt} &= -\frac{1}{L}\left(\frac{dp'}{dp}\right)_{\text{horizontal}}\cdot\frac{\delta p}{\delta x} \\ \frac{\rho'}{\rho}\rho g &= -\frac{1}{l}\left(\frac{dp'}{dp}\right)_{\text{vertical}}\cdot\frac{\delta p}{\delta z} \end{aligned}$$

The corresponding equations for the model experiment are:

$$\text{(13)}\qquad \begin{aligned} \rho\frac{du}{dt} &= -\frac{\delta p}{\delta x} \\ \rho g &= -\frac{\delta p}{\delta z} \end{aligned}$$

Since the systems (11b) and (13) must be identical, we have:

$$\text{(14)}\qquad \begin{aligned} \left(\frac{dp'}{dp}\right)_{\text{horizontal}} &= \frac{\rho'}{\rho}\frac{L^2}{T^2} \\ \left(\frac{dp'}{dp}\right)_{\text{vertical}} &= \frac{\rho'}{\rho}l \end{aligned}$$

Dynamic similarity obviously implies the condition that these two quantities be equal. We therefore obtain as a necessary condition:

$$\text{(15)}\qquad l=\frac{L^2}{T^2}$$

This equation can be regarded as a generalization of (9a). If we put $l=L$, the two conditions become identical. (15) may be written in the form

$$\text{(15b)}\qquad \frac{L}{l}=\frac{1}{\frac{L}{T^2}}$$

and is then open to a simple physical interpretation. Denoting two characteristic horizontal and vertical distances by D' and h', corresponding respectively to D and h, and a characteristic velocity by U', corresponding to U, we obtain

$$\text{(15c)}\qquad \frac{\left|\frac{h}{D}\right|}{\left|\frac{h'}{D'}\right|}=\frac{\left|\frac{dU}{dt}\right|}{\left|\frac{dU'}{dt'}\right|}$$

In this form the condition (15) denotes that the horizontal accelerations of the model movements will be magnified at the same rate as the vertical dimensions of the model are exaggerated.

The formula (15) can be applied with advantage also in comparing atmospheric systems of different vertical dimensions, especially in cases where the deviating force of the earth's rotation can be neglected. Suppose that we wish to determine the velocity with which a body of cold air, surrounded by warmer and lighter air, is dilating horizontally. Comparing two such bodies, the vertical dimensions of which have the ratio $\frac{l}{1}$, we see from (15), that the ratio between corresponding horizontal velocities is $\frac{L}{T}=\sqrt{l}$; that is, the horizontal velocity of a cold wave should be proportional to the square root of its height. Thus we find again in a more general way a result, which previously and by other means has been derived by Exner (*2*). In cases where the deviating force must be taken into consideration, the condition (15) reduces to

$$\text{(15d)}\qquad l=L^2,$$

since the time scale T now is equal to 1. This formula can be interpreted in the following way:

If we take any complete atmospheric system, a cyclone surrounded by homogeneous air at rest, for instance, and magnify the horizontal dimensions L times, the vertical dimensions l times, then the original and the new system are dynamically similar, provided $l=L^2$. In two dynamically similar atmospheric systems the ratio between corresponding horizontal velocities is equal to the square root of the ratio between corresponding vertical dimensions.

As an application and test of the condition (15) we will solve the following problem. A liquid in (A') is rotating about a vertical axis with the angular velocity Ω'. In the model vessel (A) another liquid is rotating with the angular velocity Ω. It is to be shown that the free surface in (A) can be obtained from the free surface in (A') by use of the transformation

$$\text{(16)}\qquad x'=Lx,\ y'=Ly,\ z'=lz,\ t'=Tt'$$

The free surface in (A') is a surface of constant pressure. We have

$$p' = \text{const.} - \rho' g z + \frac{\rho' \Omega'^2 (x'^2 + y'^2)}{2}$$

Thus, omitting a constant, we have at the free surface

$$z' = \frac{\Omega'^2}{2g} (x'^2 + y'^2) \tag{17}$$

For the free surface in the model space (A) we obtain in the same way

$$z = \frac{\Omega^2}{2g} (x^2 + y^2) \tag{18}$$

To show that the latter surface can be derived from the former through the transformation (16), we must first determine the time scale. Since angular velocity has the dimension of an inverse time, we have

$$\Omega' = \frac{1}{T} \Omega \tag{19}$$

Thus we obtain from (16), (17) and (19)

$$lz = \frac{\Omega^2}{2g} \frac{L^2}{T^2} (x^2 + y^2) \tag{20}$$

Since, according to (15),

$$l = \frac{L^2}{T^2}$$

the equations (20) and (18) become identical, *q. e. d.*

Making allowance for (15), we will now construct a model vessel suitable for our experiments. As is well known, atmospheric movements are to a considerable extent determined by the deviating force of the earth's rotation. Our first task is therefore to imitate this force in a convenient way. This can be done by making the experiments in a vessel, rotating about a vertical axis at an angular velocity Ω of say, n rotations per minute $\left(\Omega = \frac{2\pi n}{60}\right)$. Now the earth's surface is everywhere orthogonal to the apparent gravity. This would not be the case in a rotating vessel with plane bottom, since the rotation produces a horizontal, outwardly directed centrifugal force. The equipotential surfaces for the resultant of gravity and centrifugal force are paraboloids of the form

$$z = \frac{\Omega^2}{2g} (x^2 + y^2) \tag{21}$$

In order to produce dynamic similarity we must give the bottom of the vessel the form of an equipotential surface. For this purpose the following procedure, suggested by Professor Humphreys, may be useful. The rotating vessel is partly filled with melted paraffin, the free surface of which will gradually assume the form (*21*). Keeping the vessel in rotation until the paraffin is solidified, we obtain the bottom form desired. This method has the additional advantage that the surface form can easily be changed when a new speed of rotation is chosen.

Suppose, that the atmospheric phenomena which we intend to study take place at about 45° north latitude within a circular area of 4,000 km. diameter and are entirely restricted to the troposphere (10 km.). Using a height scale of

$$l = 25.10^4$$

and a length scale of

$$L = 2.10^6$$

we obtain for the time scale

$$T = 4.10^3$$

The portion of the atmosphere considered will then be imitated by a circular liquid layer of 4 cm. thickness and 2 m. diameter. Since the angular velocity Ω' of the earth at 45° north latitude has the value

$$\Omega' = \frac{2\pi \sin 45°}{24.60.60}$$

the number of rotations (n) may be determined from the equation

$$\frac{2\pi \sin 45°}{24.60.60} = \frac{1}{4.10^3} \cdot \frac{2\pi n}{60}$$

or

$$n = 1.96$$

The corresponding angular velocity is

$$\Omega = 0.205$$

From the values of L and T the following relation between corresponding velocities is obtained:

$$u' = \frac{L}{T} u = 5.10^2$$

Thus an atmospheric velocity of 5 m. p. s. will give a model velocity of 1 cm. p. s.

The exaggeration of the vertical dimensions in the model is given by

$$\frac{L}{l} = \frac{2.10^6}{25.10^4} = 8$$

Using the previously derived numerical value of Ω we can easily compute the elevation of the paraboloid (21) above the horizontal plane $z = O$. In a distance of 50 cm. from the axis this elevation is only 0.5 mm. and amounts at the edge of the vessel to 2 mm.

The numerical constants of the model vessel are presented in Table 1.

TABLE 1.—*Numerical constants of the model vessel*

Length scale (L)	2×10^6.
Vertical scale (l)	25×10^4.
Exaggeration of height $\left(\frac{L}{l}\right)$	8.
Time scale (T)	4×10^3.
Rotations per min. (n)	1.96.
Angular velocity (Ω)	0.205.
Velocity scale $\left(\frac{L}{T}\right)$	5×10^2.
Height of troposphere (10 km.) in model	4 cm.
Diameter of system (4,000 km.) in model	2 m.
Bottom elevation 50 cm. from axis	0.5 mm.
Bottom elevation 100 cm. from axis	2.0 mm.

LITERATURE CITED

(*1*) BAIRSTOW, L.

1920. APPLIED AERODYNAMICS, (New York.)

(*2*) DYNAMISCHE METEOROLOGIE. (Vienna, 1925), p. 319.